NUMERICAL MODELING OF JET FLOWS OF A VISCOUS LIQUID

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Flows in the jet boundary layer of an incompressible liquid are studied using a numerical finite-difference method which is developed.

Jet flows, which are widely used in various technological devices, have an essentially non-self-similar nature as a rule. This is connected with the fact that the extent of the region of flow is comparable in many cases with the characteristic size of the exit cross section of the nozzle device, and the flow is not able to reach a self-similar state. The non-self-similarity, the strong dependence of the flow on the discharge conditions, hinders the use of the well-known integral methods of calculation [1], and therefore numerical finite-difference methods are presently used for the solution of such problems. On the other hand, an undeniable advantage of finite-difference methods over integral methods is the fact that the former allow one to avoid the assumptions and simplifying hypotheses inherent to integral methods and to use more complex systems of modeling equations.

Let us consider the problem of the jet flow of a viscous incompressible liquid. In the boundary-layer approximation the system of equations of motion has the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\mathrm{Re}^{-1}\frac{1}{y^k}\frac{\partial}{\partial y}y^k\frac{\partial u}{\partial y}-\frac{1}{y^k}\frac{\partial}{\partial y}y^k\tau;$$
$$\frac{\partial u}{\partial x}+\frac{1}{y^k}\frac{\partial}{\partial y}y^k v=0; \quad (1)$$

k = 1 for axisymmetric flow and k = 0 for plane flow. For closure of the system of equations of motion (1) we use the Kolmogorov–Prandtl model of turbulence [2, 3], in accordance with which the turbulent viscosity ε, determined as

$$\varepsilon=-\tau/\frac{\partial u}{\partial y},$$

from considerations of dimensionality is expressed in the form

$$\varepsilon=c_\varepsilon q\Lambda. \quad (2)$$

To determine $\overline{q}^2$ we use the equation of conservation of the kinetic energy of turbulent pulsations [4]

$$u\frac{\partial\overline{q^2}}{\partial x}+v\frac{\partial\overline{q^2}}{\partial y}=-\frac{1}{y^k}\frac{\partial}{\partial y}y^k\overline{v'(q^{2'}+p'/\rho)}-\tau\frac{\partial u}{\partial y}-D,$$

in which the dissipation of turbulent energy is determined from the Kolmogorov equation [2]

$$D=c_D q^3/\Lambda,$$

while the term $v'(q^{2'}+p'/\rho)$, determining the transverse transport of pulsation energy by the pulsating motion, is represented in the diffusion form [3]:

$$\overline{v'(q^{2'}+p'/\rho)}=-\frac{\varepsilon}{c_q}\frac{\partial\overline{q^2}}{\partial y}.$$

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We take the scale of turbulence Λ as proportional to the transverse integral scale of turbulence, which on the basis of an analysis of the results of measurements [5-8] can be considered as constant in a cross section of a jet and linearly dependent on the longitudinal coordinate. Then

$$\Lambda = c_\Lambda x. \tag{3}$$

Thus, we have a closed system of Eqs. (2)-(4) describing the flow in the jet boundary layer of an incompressible viscous liquid:

$$\begin{gathered} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{y^k}\frac{\partial}{\partial y} y^k(\varepsilon + \mathrm{Re}^{-1})\frac{\partial u}{\partial y}; \\ \frac{\partial u}{\partial x} + \frac{1}{y^k}\frac{\partial}{\partial y} y^k v = 0; \\ u\frac{\partial \overline{q}^2}{\partial x} + v\frac{\partial \overline{q}^2}{\partial y} = \frac{1}{c_q}\frac{1}{y^k}\frac{\partial}{\partial y} y^k \varepsilon \frac{\partial \overline{q}^2}{\partial y} + \varepsilon\left(\frac{\partial u}{\partial y}\right)^2 \quad c_D q^3/\Lambda. \end{gathered} \tag{4}$$

We take c_ε, c_D, c_Λ, and c_q as constant in the entire field of flow.

For the numerical solution of the system obtained it is necessary to approximate it by difference equations at the nodes of a finite-difference grid. In the given case the use of the most common rectangular grid in the (x, y) plane proves to be ineffective, since the thickness of the jet increases strongly in the direction of flow, which leads to a continuous increase in the number of grid nodes analyzed. The transformation

$$\eta = \frac{y}{x + x^*}\xi$$

of the transverse coordinate allows one to decrease the growth of the boundary layer of the jet or to eliminate it entirely. Here the quantities x^* and ξ are determined at the start of the calculation on the basis of preliminary estimates of the development of the jet boundary layer. At the same time, they can be corrected in the process of calculation on the basis of the information obtained.

Applying the transformation described to system (4), we obtain in the (x, η) plane

$$\begin{gathered} u\frac{\partial u}{\partial x} + \frac{1}{x + x^*}(\xi v - u\eta)\frac{\partial u}{\partial \eta} = \frac{\xi^2}{(x + x^*)^2}\frac{1}{\eta^k}\frac{\partial}{\partial \eta}\eta^k(\varepsilon + \mathrm{Re}^{-1})\frac{\partial u}{\partial \eta}; \\ \frac{\partial u}{\partial x} - \frac{\eta}{x + x^*}\frac{\partial u}{\partial \eta} + \frac{\xi}{x + x^*}\frac{1}{\eta^k}\frac{\partial}{\partial \eta}\eta^k v = 0; \\ u\frac{\partial \overline{q}^2}{\partial x} + \frac{1}{x + x^*}(\xi v - u\eta)\frac{\partial \overline{q}^2}{\partial \eta} = \frac{\xi^2}{(x + x^*)^2}\frac{1}{c_q}\frac{1}{\eta^k}\frac{\partial}{\partial \eta}\eta^k\varepsilon\frac{\partial \overline{q}^2}{\partial \eta} + \frac{\xi^2}{(x + x^*)^2}\varepsilon\left(\frac{\partial u}{\partial \eta}\right)^2 - c_D q^3/\Lambda. \end{gathered} \tag{5}$$

The system of Eqs. (5) must be supplemented with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial \overline{q}^2}{\partial \eta} = v = 0 \quad \text{at} \quad \eta = 0;$$

$$u = U_1, \quad \overline{q}^2 = Q \quad \text{at} \quad \eta = \eta_M.$$

The proposed numerical algorithm for the solution is based on the idea of scaling. The preliminary values of the unknown functions are initially found at the half-step of the difference grid using the values of the functions from the preceding layer as the coefficients of the difference equations. The preliminary values found are then used for the whole step. This allows one to avoid iterations due to the nonlinearity of the initial differential equations, retaining a second order of approximation.

Let us describe the algorithm for the solution in more detail. The differential equations are represented in finite-difference form using an implicit, two-layer, six-point system on the grid

$$x_n = n\Delta x; \quad \eta_m = m\Delta\eta; \quad n = 0, 1, \ldots; \quad m = 0, 1, \ldots, M.$$

In this case the central differences are used for the first derivatives and the usual three-point equation is used for the second derivatives. Suppose the values of all the unknown functions are known at the layer $x = x_n$. We determine $u_{n+1/2}$ and $\overline{q}^2_{n+1/2}$ using the systems of difference equations

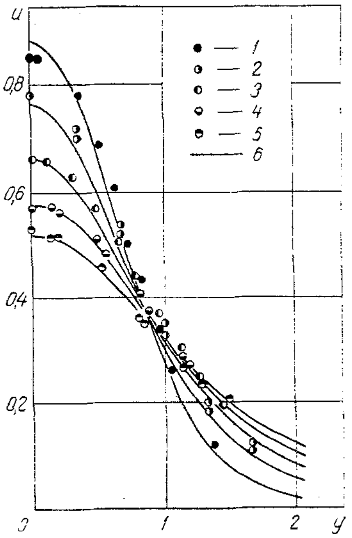

Fig. 1

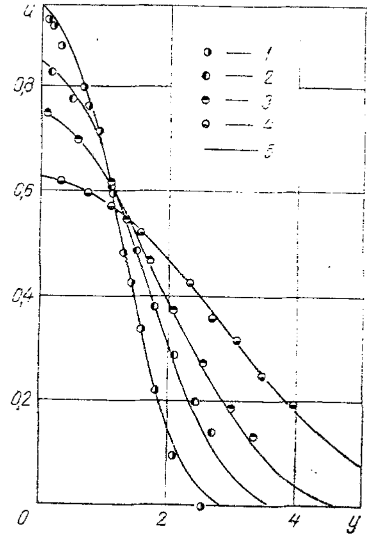

Fig. 2

Fig. 1. Development of the longitudinal velocity profile in a round laminar jet which initially has a parabolic velocity profile: 1-5) experimental data [13]. [1) x = 8; 2) 16; 3) 24; 4) 32; 5) 40]; 6) calculation.

Fig. 2. Mixing region of a plane turbulent jet. Development of longitudinal velocity profile: 1-4) experimental data [14]. [1) x = 10; 2) 15; 3) 20; 4) 30]; 5) calculation.

$$A_{n+\frac{1}{2},m}\, u_{n+\frac{1}{2},m-1} + B_{n+\frac{1}{2},m}\, u_{n+\frac{1}{2},m} + C_{n+\frac{1}{2},m}\, u_{n+\frac{1}{2},m+1} = D_{n+\frac{1}{2},m}; \tag{6a}$$

$$E_{n+\frac{1}{2},m}\, \overline{q^2}_{n+\frac{1}{2},m-1} + F_{n+\frac{1}{2},m}\, \overline{q^2}_{n+\frac{1}{2},m} + G_{n+\frac{1}{2},m}\, \overline{q^2}_{n+\frac{1}{2},m+1} = H_{n+\frac{1}{2},m}, \quad m = 1, 2, \ldots, M-1, \tag{6b}$$

where

$$A_{n+\frac{1}{2},m} = -a - b; \quad B_{n+\frac{1}{2},m} = b + c + d; \quad C_{n+\frac{1}{2},m} = a - c;$$

$$D_{n+\frac{1}{2},m} = du_{p,m} - a\,(u_{n,m+1} - u_{n,m-1}) + c\,(u_{n,m+1} - u_{n,m}) - b\,(u_{n,m} - u_{n,m-1});$$

$$E_{n+\frac{1}{2},m} = -a - b/c_q; \quad F_{n+\frac{1}{2},m} = (b + c)/c_q + d;$$

$$G_{n+\frac{1}{2},m} = a - c/c_q; \quad H_{n+\frac{1}{2},m} = d\overline{q}^2_{p,m} - a\,(\overline{q}^2_{n,m+1} - \overline{q}^2_{n,m-1})$$

$$+ [c\,(\overline{q}^2_{n,m+1} - \overline{q}^2_{n,m}) - b\,(\overline{q}^2_{n,m} - \overline{q}^2_{n,m-1})]/c_q + f\varepsilon_{p,m}\,(u_{p,m+1} - u_{p,m-1})^2 - c_D\,(q_{p,m})^3\,\Delta\eta/\Lambda_p;$$

$$a = 0.25\,(\xi v_{p,m} - u_{p,m}\eta_m)/(x_l + x^*);$$

$$b = (2\varkappa\,\mathrm{Re}^{-1} + \varepsilon_{p,m} + \varepsilon_{p,m-1})\,(\eta_{m-\frac{1}{2}}/\eta_m)^k f;$$

$$c = (2\varkappa\,\mathrm{Re}^{-1} + \varepsilon_{p,m} + \varepsilon_{p,m+1})\,(\eta_{m+\frac{1}{2}}/\eta_m)^k f;$$

$$d = u_{n,m}\Delta\eta/t; \quad f = 0.25\,[\xi/(x_l + x^*)]^2/\Delta\eta; \quad t = \frac{1}{2}\,\Delta x; \quad p = n;$$

$$l = n + \frac{1}{4};$$

$\varkappa = 1$ for the equation of motion and $\varkappa = 0$ for the equation of balance of turbulent kinetic energy.

The systems (6a) and (6b) together with the boundary conditions

$$u_{n+\frac{1}{2},1} = u_{n+\frac{1}{2},0}; \quad \overline{q}^2_{n+\frac{1}{2},1} = \overline{q}^2_{n+\frac{1}{2},0}; \quad u_{n+\frac{1}{2},M} = U_1;$$

$$\bar{q}^2_{n+\frac{1}{2}, M} = Q \tag{7}$$

are solved by the trial-run method.

As was shown in [10], in the case of the calculation of flooded jet flows ($U_1 = 0$) the necessary condition of good conditionality [9] is not satisfied for the boundary difference problem (6a), (6b), (7), which can lead to computational instability of the trial-run method. In such a case one can use, for example, a variant of the trial-run method – the nonmonotonic trial-run method [11]. Following this method one determines the transverse velocity from the difference analog of the continuity equation

$$v_{n+\frac{1}{2}, m} = \left(\frac{\eta_m}{\eta_{m+1}}\right)^k v_{n+\frac{1}{2}, m} + \left(\frac{\eta_{m+\frac{1}{2}}}{\eta_{m+1}}\right)^k \frac{1}{\xi}$$

$$\times [-(x_l + x^*)(u_{n+\frac{1}{2}, m} + u_{n+\frac{1}{2}, m+1} - u_{n,m} - u_{n,m+1}) \frac{\Delta\eta}{2t}$$

$$+ \frac{1}{2} \eta_{m+\frac{1}{2}} (u_{n+\frac{1}{2}, m+1} + u_{n,m+1} - u_{n+\frac{1}{2}, m} - u_{n,m})]; \quad v_{n+\frac{1}{2}, 0} = 0; \tag{8}$$

the scale of turbulence $\Lambda_{n+1/2}$ and the turbulent viscosity $\varepsilon_{n+1/2}$ are determined using the corresponding equations. The next step will be the calculation of u_{n+1} and $\bar{q}^2_{n+1}$ in accordance with the systems of difference equations

$$A_{n+1,m} u_{n+1,m-1} + B_{n+1,m} u_{n+1,m} + C_{n+1,m} u_{n+1,m+1} = D_{n+1,m};$$

$$E_{n+1,m} \bar{q}^2_{n+1,m-1} + F_{n+1,m} \bar{q}^2_{n+1,m} + G_{n+1,m} \bar{q}^2_{n+1,m+1} = H_{n+1,m};$$

$$m = 1, 2, \ldots, M-1,$$

where the coefficients $A_{n+1,m}$, $B_{n+1,m}, \ldots, H_{n+1,m}$ are analogous to the corresponding coefficients $A_{n+1/2,m}$, $B_{n+1/2,m}, \ldots, H_{n+1/2,m}$, in which $t = \Delta x$, while the indices p and l are replaced by n + 1. Then having determined the transverse velocity v_{n+1} from Eq. (8), replacing the index n + ½ by n + 1 and setting $t = \Delta x$ and $l = n + 1/2$ in it, the scale Λ_{n+1}, and the viscosity ε_{n+1} one can end the cycle.

The values of the unknown functions at the "zeroth" layer x_0 are given by the initial conditions, where the transverse velocity can either be taken equal to 0 or found by the means used in the numerical continuation method [12].

The method presented was realized on an M-222 computer, with the calculation time for one axial cross section using 50 node points across the jet being on the order of 0.5 sec. The accuracy of the method and the applicability of the turbulence model used were estimated through a comparison of the results obtained with analogous numerical and analytical solutions, as well as with certain experimental data.

The calculated development of the axial velocity profile in a flooded round laminar jet which initially has a parabolic velocity profile is presented in Fig. 1. The results of the calculation agree well with the experimental data of [13], with the velocity profile practically coinciding with the classical Schlichting solution [12] at a considerable distance from the nozzle cut. Here it should be noted that in the case of laminar flows the equations of motion are exact (with allowance for the fact that the concept of the boundary layer is only an approximation of the true physical phenomenon, of course) and are closed without the inclusion of semiempirical equations, and therefore calculations of such flows give a good estimate of the applicability and accuracy of the numerical method itself.

Figure 2 illustrates the calculation of a turbulent jet in the mixing region. The constants entering into Eqs. (2), (3), and (4) had the following values: $c_\varepsilon = 1$; $c_D = 0.09$; $c_\Lambda = 0.014$; and $c_q = 1$. The experimental data are taken from [14].

The numerical method presented for the modeling of mixing in jets of incompressible viscous liquid can be generalized without difficulty to flows with rotation, compressible flows, and to problems with heat and mass transfer.

NOTATION

x, y	are the longitudinal and transverse coordinates, respectively, normalized to half-width (radius) of nozzle;
u, v	are the average longitudinal and transverse velocities normalized to characteristic velocity U_0 at jet orifice;

τ, q^2	are the turbulent shear stress and kinetic energy of turbulent pulsations normalized to U_0^2;
$q = \sqrt{q^2}$; ρ	is the density;
p	is the pressure normalized to ρU_0^2;
Λ, U_1, Q	are the linear scale of turbulence, velocity of outer flow, and pulsation energy of outer flow normalized to half-width (radius) of nozzle;
U_0, U_0^2, c_ε, c_D, c_q, c_Λ, x^*, ξ	are the parameters of turbulence model;
x^*, ξ	are the parameters of coordinate transformation;
Δx, $\Delta \eta$	are the grid steps;
η_M	is the value of coordinate η corresponding to boundary of jet.

Indices

' is the pulsation component;

— is the averaged component.

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